Assignment 8 Senior Challenging Counting and Probability Questions

21. Box 1 contains one gold marble and one black marble. Box 2 contains one gold marble and two black marbles. Box 3 contains one gold marble and three black marbles. Whenever a marble is chosen randomly from one of the boxes, each marble in that box is equally likely to be chosen. A marble is randomly chosen from Box 1 and placed in Box 2. Then a marble is randomly chosen from Box 2 and placed in Box 3. Finally, a marble is randomly chosen from Box 3. What is the probability that the marble chosen from Box 3 is gold?

(A) $\frac{11}{40}$ **(B)** $\frac{3}{10}$ **(C)** $\frac{13}{40}$ **(D)** $\frac{7}{20}$ **(E)** $\frac{3}{8}$

24. Eight teams compete in a tournament. Each pair of teams plays exactly one game against each other. There are no ties. If the two possible outcomes of each game are equally likely, what is the probability that every team loses at least one game and wins at least one game?

(A) $\frac{1799}{2048}$ **(B)** $\frac{1831}{2048}$ **(C)** $\frac{1793}{2048}$ **(D)** $\frac{903}{1024}$ **(E)** $\frac{889}{1024}$

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21. Amina and Bert alternate turns tossing a fair coin. Amina goes first and each player takes three turns. The first player to toss a tail wins. If neither Amina nor Bert tosses a tail, then neither wins. What is the probability that Amina wins?

(A) $\frac{21}{32}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{3}{7}$ **(D)** $\frac{11}{16}$ **(E)** $\frac{9}{16}$

Problem 1. Alan and Beti alternately toss a fair die, with Alan going first. A neutral third party keeps a running tab of the combined sum of all their throws. Whoever first reaches a combined sum divisible by 6 wins. What is the probability that Alan wins?

9. (a) The string $AAABBBAABB$ is a string of ten letters, each of which is A or B, that does not include the consecutive letters $ABBA$.
The string $AAABBAAABB$ is a string of ten letters, each of which is A or B, that does include the consecutive letters $ABBA$.
Determine, with justification, the total number of strings of ten letters, each of which is A or B, that do not include the consecutive letters $ABBA$.

Problem 6

Melinda has three empty boxes and 12 textbooks, three of which are mathematics textbooks. One box will hold any three of her textbooks, one will hold any four of her textbooks, and one will hold any five of her textbooks. If Melinda packs her textbooks into these boxes in random order, the probability that all three mathematics textbooks end up in the same box can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

9. A *permutation* of a list of numbers is an ordered arrangement of the numbers in that list. For example, $3, 2, 4, 1, 6, 5$ is a permutation of $1, 2, 3, 4, 5, 6$. We can write this permutation as $a_1, a_2, a_3, a_4, a_5, a_6$, where $a_1 = 3, a_2 = 2, a_3 = 4, a_4 = 1, a_5 = 6$, and $a_6 = 5$.

(a) Determine the average value of

$$|a_1 - a_2| + |a_3 - a_4|$$

over all permutations a_1, a_2, a_3, a_4 of $1, 2, 3, 4$.

(b) Determine the average value of

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7$$

over all permutations $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ of $1, 2, 3, 4, 5, 6, 7$.

(c) Determine the average value of

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_{197} - a_{198}| + |a_{199} - a_{200}| \qquad (*)$$

over all permutations $a_1, a_2, a_3, ..., a_{199}, a_{200}$ of $1, 2, 3, 4, \ldots, 199, 200$. (The sum labelled $(*)$ contains 100 terms of the form $|a_{2k-1} - a_{2k}|$.)

10. For each positive integer $n \geq 1$, let C_n be the set containing the n smallest positive integers; that is, $C_n = \{1, 2, \ldots, n-1, n\}$. For example, $C_4 = \{1, 2, 3, 4\}$. We call a set, F, of subsets of C_n a *Furoni family of* C_n if no element of F is a subset of another element of F.

(a) Consider $A = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$. Note that A is a Furoni family of C_4. Determine the two Furoni families of C_4 that contain all of the elements of A and to which no other subsets of C_4 can be added to form a new (larger) Furoni family.

(b) Suppose that n is a positive integer and that F is a Furoni family of C_n. For each non-negative integer k, define a_k to be the number of elements of F that contain exactly k integers. Prove that

$$\frac{a_0}{\binom{n}{0}} + \frac{a_1}{\binom{n}{1}} + \frac{a_2}{\binom{n}{2}} + \cdots + \frac{a_{n-1}}{\binom{n}{n-1}} + \frac{a_n}{\binom{n}{n}} \leq 1$$

(The sum on the left side includes $n + 1$ terms.)

(Note: If n is a positive integer and k is an integer with $0 \leq k \leq n$, then $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of subsets of C_n that contain exactly k integers, where $0! = 1$ and, if m is a positive integer, $m!$ represents the product of the integers from 1 to m, inclusive.)

(c) For each positive integer n, determine, with proof, the number of elements in the largest Furoni family of C_n (that is, the number of elements in the Furoni family that contains the maximum possible number of subsets of C_n).

Problem 1. Alan and Beti alternately toss a fair die, with Alan going first. A neutral third party keeps a running tab of the combined sum of all their throws. Whoever first reaches a combined sum divisible by 6 wins. What is the probability that Alan wins?

Solution. Suppose that the current sum is not divisible by 6. What is the probability that the next toss makes it divisible by 6? It is clear that there is exactly one toss that will do the job. For instance, if the current sum leaves a remainder of 4 on division by 6, then only a 2 will make the next sum divisible by 6. So if the current sum is not divisible by 6, the probability that the next sum is divisible by 6 is $1/6$.

Now we compute the probability that Alan wins. There are many ways this could happen.

(i) The game could last exactly one toss. The probability of this is $1/6$.

(ii) The game could last exactly 3 tosses. The probability of this is $(5/6)(5/6)(1/6)$.

(iii) The game could last exactly 5 tosses. The probability of this is $(5/6)^4(1/6)$.

And so on, forever! The probability that Alan wins is therefore

$$\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)+\cdots.$$

That is an infinite geometric series with first term $1/6$ and common ratio $(5/6)^2$. There is a standard formula for the sum.

Another Way. Let p be the probability that Alan wins. Alan can win in two ways: (i) immediately and (ii) later. The probability that Alan wins immediately is $1/6$. Thus the probability he does not win immediately (and possibly ultimately loses) is $5/6$.

If Alan does not win immediately, Beti now is in essence "starting," so the probability she wins ultimately is p, and therefore the probability Alan ultimately wins is $1-p$. It follows that

$$p=\frac{1}{6}+\frac{5}{6}(1-p).$$

We now have a linear equation for p. Solve. We get $p=6/11$.

Solution One

The total ways the textbooks can be arranged in the 3 boxes is $12\textbf{C}3\cdot 9\textbf{C}4$, which is equivalent to $\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{144}=12\cdot 11\cdot 10\cdot 7\cdot 3$. If all of the math textbooks are put into the box that can hold 3 textbooks, there are $9!/(4!\cdot 5!)=9\textbf{C}4$ ways for the other textbooks to be arranged. If all of the matt textbooks are put into the box that can hold 4 textbooks, there are 9 ways to choose the other book in that box, times $8\textbf{C}3$ ways for the other books to be arranged. If all of the math textbooks are put into the box with the capability of holding 5 textbooks, there are $9\textbf{C}2$ ways to choose the other 2 textbooks in that box, times $7\textbf{C}3$ ways to arrange the other 7 textbooks. $9\textbf{C}4=9\cdot 7\cdot 2=126$, $9\cdot 8\textbf{C}3=9\cdot 8\cdot 7=504$, and $9\textbf{C}2\cdot 7\textbf{C}3=9\cdot 7\cdot 5\cdot 4=1260$, so the total number of ways the math textbooks can all be placed into the same box is $126+504+1260=1890$. So, the probability of this occurring is $\frac{(9\cdot 7)(2+8+(4\cdot 5))}{12\cdot 11\cdot 10\cdot 7\cdot 3}=\frac{1890}{27720}$. If the numerator and denominator are both divided by $9\cdot 7$, we have $\frac{(2+8+(4\cdot 5))}{4\cdot 11\cdot 10}=\frac{30}{440}$. Simplifying the numerator yields $\frac{30}{10\cdot 4\cdot 11}$, and dividing both numerator and denominator by 10 results in $\frac{3}{44}$. This fraction cannot be simplified any further, so $m=3$ and $n=44$. Therefore, $m+n=3+44=\boxed{047}$.